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# Canonical Fractional-Step Methods and Consistent Boundary Conditions for the Incompressible Navier–Stokes Equations

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An account of second-order fractional-step methods and boundary conditions for the incompressible Navier–Stokes equations is presented. The goals of the work were (i) identification and analysis of all possible splitting methods of second-order splitting accuracy, and (ii) determination of consistent boundary conditions that yield second-order-accurate solutions. Exact and approximate block-factorization techniques were used to construct second-order splitting methods. It has been found that only three canonical types (D, P, and M) of splitting methods are nondegenerate, and all other second-order splitting schemes are either degenerate or equivalent to them. Investigation of the properties of the canonical methods indicates that a method of type D is recommended for computations in which zero divergence is preferred, while a method of type P is better suited to cases where highly accurate pressure is more desirable. The consistent boundary conditions on the tentative velocity and pressure have been determined by a procedure that consists of approximation of the split equations and the boundary limit of the result. It has been found that the pressure boundary condition is independent of the type of fractional-step methods. The consistent boundary conditions on the tentative velocity were determined in terms of the natural boundary condition and derivatives of quantities available at the current time step (to be evaluated by extrapolation). Second-order fractional-step methods that admit the zero-pressure-gradient boundary condition have been derived by using a transformation that involves the "delta form" pressure. The boundary condition on the new tentative velocity becomes greatly simplified due to improved accuracy built into the transformation. 

*Key Words:* Navier–Stokes equations; fractional-step methods; approximate factorization; boundary conditions; incompressibility.



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## 1. INTRODUCTION

This paper is concerned with fractional-step methods and boundary conditions for the Navier–Stokes equations that describe the three-dimensional, unsteady incompressible flow in primitive variables. Numerical solution of the discretized Navier–Stokes equations is expensive in general, because the resulting algebraic equations for the velocity components and pressure are coupled and hence form a very large system. This difficulty can be alleviated by employing a fractional-step method in which the coupled system is split into decoupled systems of much smaller size that can be solved with much less computational cost [4, 6, 11, 12].

There are two important issues that need to be properly accounted for in the construction and implementation of a fractional-step method: (i) splitting (or decoupling) of the equations (or operators); and (ii) boundary conditions associated with the tentative velocity and pressure. These issues have been subject to controversy in the literature primarily because they are not independent of each other and involve implementation details such as discretization schemes (both time and space) and configuration of computational domain. The accuracy of the solutions obtained by a fractional-step method depends on the accuracy of the time-advancement method, splitting method, and boundary conditions, which are referred to hereinafter as the *time accuracy*, *splitting accuracy*, and *accuracy of the boundary conditions*, respectively. The same accuracy of splitting and boundary conditions for a fractional-step method is guaranteed only if they are *consistent* with the discretized Navier–Stokes equations and natural boundary conditions to the order of accuracy of the time-advancement method.

The purpose of the present work is twofold: (i) to identify and analyze all possible splitting methods of second-order splitting accuracy; and (ii) to determine consistent boundary conditions that yield second-order-accurate solutions. The account given here is straightforward and the results do not depend on a particular discretization scheme for time and spatial derivatives or the geometry of the computational domain.

It was shown by the recent studies of Dukowicz and Dvinsky [5] and Perot [10] that splitting in a fractional-step method can be regarded as approximate block-factorization of the discretized equations. The splitting accuracy of equations obtained by *ad hoc* splitting may or may not be of the same order of time accuracy. However, splitting based on approximate factorization always guarantees that the resulting system of equations is *consistent* with the unsplit equations to the order of accuracy of the time-advancement method; i.e., the splitting accuracy is of the same order of the time accuracy. This distinguished feature of approximate factorization is used to construct fractional-step methods of second-order accuracy in this work. The procedure consists of two stages: exact and approximate block-factorizations. All possible splittings of the discretized Navier–Stokes equations have been derived by exact block-factorization (developed in Section 3). The results have been approximated to second-order accuracy and analyzed to determine three non-degenerate fractional-step methods referred to as the *canonical methods* of type D, P, and M, respectively (see Section 4). Notice that all other fractional-step methods developed on the basis of approximate factorization are either degenerate or equivalent to one of the canonical methods. The present account can be considered as a generalization of the view held in Dukowicz and Dvinsky [5] and Perot [10].

The solutions to a canonical fractional-step method of second-order accuracy are not guaranteed to be second-order accurate, if the boundary conditions for the split equations are not specified properly. The boundary conditions on the nonphysical quantities in the

split equations cannot be specified *a priori*, and hence the crux of the determination of the boundary conditions is to express them in terms of known quantities available at the current time step. It is desired to have boundary conditions that are *consistent* with the split equations of the fractional-step methods. By consistency of boundary conditions, it is meant that the boundary conditions satisfy the split equations at the boundary to the order of the splitting accuracy or equivalently the time accuracy. The consistent boundary conditions can be identified by the limiting process of the equations obtained by consistent approximation (see Section 6). It has been found that the procedure developed in this work can be used as a powerful tool in finding the consistent boundary condition for a fractional-step method.

There has been a certain degree of ambiguity and controversy in how the boundary conditions on the pressure are implemented in practice (see Section 6). In particular, application of the homogeneous Neumann boundary condition on the pressure leads to results of first-order accuracy only. Second-order fractional-step methods that admit the homogeneous Neumann boundary condition have been developed by transforming the solution variables and right-hand sides of the equations of the canonical methods (see Section 7). It is shown that the "delta-form" pressure, which appeared previously in Van Kan [13] and Dukowicz and Dvinsky [5], can in fact be derived from the transformation. Introduction of the delta-form pressure yields improvement of the accuracy of the tentative velocity to $O(\Delta t^2)$, leading to the extremely simple boundary condition on the tentative velocity. The role played by the delta-form pressure in the projection into the divergence-free subspace is examined here in the light of Chorin's decomposition idea [4].

This paper is organized as follows. The discretized formulation of the Navier–Stokes equations is discussed in Section 2. In Section 3, a procedure by which exact block-factorization of the discretized Navier–Stokes equations is carried out is explained. In Section 4, second-order splitting methods are constructed by using approximate block-factorization. The results are analyzed to identify three canonical second-order methods. In Section 5, characteristics of the canonical methods are investigated, including splitting and projection. Consistent boundary conditions on the tentative velocity and pressure are derived in general form and the case of the steady zero-velocity boundary is discussed in Section 6. In Section 7, the canonical methods are transformed such that the homogeneous Neumann boundary conditions are satisfied by the pressure, and the consistent boundary conditions for the new fractional-step methods are discussed. Concluding remarks are presented in Section 8.

## 2. DISCRETIZED FORMULATION OF THE NAVIER–STOKES EQUATIONS

The incompressible Navier–Stokes equations, which consist of the momentum equations and continuity equation, can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u}, \tag{2.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1b}$$

where $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ are the velocity vector and pressure, respectively. The momentum and continuity equations have been made dimensionless by an appropriate reference velocity and length scale and Re denotes the Reynolds number of the flow under consideration. In this work, the initial velocity field is assumed to be divergence-free. The boundary conditions

on the velocity components are usually specified on the boundary at $\mathbf{x}_B$:

$$\mathbf{u}_B(t) = \mathbf{u}(\mathbf{x}_B, t). \tag{2.2}$$

Since it is difficult to know the boundary condition on the pressure *a priori* in most practical computations, it is a common practice to derive them from appropriate considerations (rather than specify them).

The major difficulty in obtaining a time-accurate numerical solution for an incompressible flow arises from the fact that the time derivative does not appear explicitly in the continuity equation (2.1b). The continuity equation plays a role of kinematic constraint on the velocity vector field via the implicit coupling between the pressure and velocity fields. Since the Navier–Stokes equations are a coupled system, it is important to develop simplified and efficiently solvable numerical approximations. This can be achieved by decoupling the pressure computation from that of the velocity, thereby reducing the problem to a system of separately-solvable equations.

When the Navier–Stokes equations (2.1) are discretized only in time by appropriate second-order time-advancement methods, the semi-discrete equations can be written as

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t \mathbb{G} p^{(n+1)} = \mathbf{r}, \tag{2.3a}$$

$$\mathbb{D}\mathbf{u}^{(n+1)} = 0, \tag{2.3b}$$

where $\mathbb{A}$, $\mathbb{G} = \text{grad}$ and $\mathbb{D} = \text{div}$ are the *continuous* differential operators and the right-hand side vector $\mathbf{r}$ contains all the quantities known at the current time step, $t_n$. The solutions $\mathbf{u}^{(n+1)}$ and $p^{(n+1)}$ of the semidiscrete equations (2.3) are regarded as functions of the space variable. The differential operator $\mathbb{A}$ of a second-order method can be written as

$$\mathbb{A} = 1 - \Delta t \mathbb{B}, \tag{2.4}$$

regardless of the choice of the time-advancement scheme(s). If the second-order Crank–Nicolson method is used in a semi-implicit scheme, the differential operator $\mathbb{B}$ takes the form of

$$\mathbb{B} = \frac{1}{2\text{Re}}\mathbb{L}, \tag{2.5}$$

where $\mathbb{L} = \nabla^2$ represents the Laplacian operator. In the case of a fully explicit scheme, $\mathbb{B} = 0$ and $\mathbb{A} = 1$. The natural boundary condition for the discretized equations can be expressed as

$$\mathbf{u}_B^{(n+1)} = \mathbf{u}(\mathbf{x}_B, t_{n+1}). \tag{2.6}$$

Equations in semidiscrete form are useful in analyzing a fractional-step method in general and in finding its boundary conditions in particular. The consistent boundary conditions that are independent of the discretization scheme for spatial derivatives can be found only if semidiscrete equations are considered.

When the Navier–Stokes equations (2.1) are fully discretized by appropriate discretization schemes for spatial derivatives and second-order methods for time advancement, the resulting system of algebraic equations can be written in matrix–vector form as

$$\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ \mathsf{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(n+1)} \\ p^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{2.7}$$

where the solution vector consists of velocity components $\mathbf{u}^{(n+1)}$ and pressure $p^{(n+1)}$ at the interior gridpoints. The block matrix of algebraic equations (2.7) denoted by

$$\mathbf{\Gamma}_{\mathrm{NS}} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ \mathsf{D} & 0 \end{bmatrix} \tag{2.8}$$

shall be referred to as the *system matrix* hereinafter. It has three nonzero submatrices representing discretization of the respective spatial derivatives.

The submatrix $\mathsf{A}$ is banded in most cases because it is desirable to treat viscous terms by an implicit method in order to avoid a highly stiff system; when all the terms in the momentum equations are treated by an explicit scheme, $\mathsf{A}$ becomes diagonal. The submatrix $\mathsf{G}$ represents the effect of the pressure force and $\mathsf{D}$ denotes the discretized divergence operator in the continuity equation. These submatrices $\mathsf{A}$, $\mathsf{G}$, and $\mathsf{D}$ are discrete equivalents of the differential operators $\mathbb{A}$, $\mathbb{G}$, and $\mathbb{D}$, respectively. The first term $\mathbf{r}$ on the right-hand side represents all the quantities in the interior of the domain known at the current time step, $t_n$. The *boundary terms*, $\mathbf{b}$ and $c$, are the discretized equivalent of the natural boundary conditions specified by the problem formulation.

The submatrix $\mathsf{A}$ represents advancement of the momentum equations (2.7) in time and can be written as

$$\mathsf{A} = \mathsf{I} - \Delta t\mathsf{B}, \tag{2.9}$$

regardless of the particular choice of the time-advancement scheme(s) used. When $\mathsf{A}$ takes the form of $\mathsf{M} - \Delta t\mathsf{B}$ (e.g., in finite-element discretization, $\mathsf{M}$ represents the mass matrix), it can always be transformed into the form (2.9). If a fully-explicit scheme is used, $\mathsf{B} = 0$ and $\mathsf{A}$ becomes $\mathsf{A} = \mathsf{I}$, where $\mathsf{I}$ is the identity matrix. In a semi-implicit scheme where the second-order Crank–Nicolson method is used for the viscous terms, $\mathsf{B}$ is given by

$$\mathsf{B} = \frac{1}{2\mathrm{Re}}\mathsf{L}, \tag{2.10}$$

where $\mathsf{L}$ denotes the discrete Laplace operator; and in the case of a fully implicit scheme with the second-order Crank–Nicolson method for the viscous and nonlinear terms, it takes the form of

$$\mathsf{B} = \frac{1}{2\mathrm{Re}}\mathsf{L} + \frac{1}{2}\mathsf{N}, \tag{2.11}$$

where $\mathsf{N}$ represents the discrete operator involving implicit treatment of the nonlinear terms.

A typical fractional-step method that splits the original equations (2.7) into two parts can be written in approximate-factorization form as

$$\begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & -\Delta t\mathsf{DG} \end{bmatrix} \begin{bmatrix} \mathbf{u}^* \\ \phi^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ c_1 \end{bmatrix}, \tag{2.12a}$$

$$\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(n+1)} \\ \phi^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^* \\ \phi^{(n+1)} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_2 \\ c_2 \end{bmatrix}, \tag{2.12b}$$

where $\mathbf{u}^*$ is called the *tentative velocity* and $\phi^{(n+1)}$ is called the *fictitious pressure*. These quantities introduced due to the splitting are not considered to be physical. The values of the intermediate boundary terms, $\mathbf{b}_1$, $\mathbf{b}_2$, $c_1$, and $c_2$, of the split parts are to be determined in order to solve the split systems of equations (2.12). Properties of this method and the

boundary conditions are discussed in detail in Section 5. The solution procedure consists of the three basic operations: (i) advancement of velocity by nonlinear and viscous terms in (2.12a); (ii) computation of the pressure $\phi$ by solving the Poisson equation in (2.12a); and (iii) update of velocity with the computed pressure (or pressure correction) in (2.12b). Note that the first two operations are represented by the block-lower-triangular matrix and the velocity update is represented by the block-upper-triangular matrix. It can be easily shown that the system matrix of equations assembled from the split equations is in fact a second-order approximation to the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ of the original equations (2.7).

One of the objectives of the present work is to *derive* all possible second-order splitting methods based on approximate factorization of the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$. To achieve the goal, a systematic procedure is employed in two stages: exact block-factorization and approximate block-factorization. Exact block-factorization of the system matrix yields 19 independent two-, three-, and four-part factorizations. Splitting methods of second-order accuracy are obtained by approximating the results to $O(\Delta t^2)$ and discarding those which cannot be implemented. Examination of the remaining legitimate methods reveals that only three splitting methods are considered to be independent: two two-part methods and one three-part method.

## 3. EXACT BLOCK-FACTORIZATION OF THE SYSTEM MATRIX

### 3.1. *Method of Exact Block-Factorization*

Exact factorization is the crucial step toward constructing fractional-step methods that have the specified splitting accuracy. The most general block-factorization of the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ of the discretized Navier–Stokes equations (2.7) involves the following four factors: a unit block-lower-triangular matrix $L$, unit block-upper-triangular matrix $U$, and two one-sided block-diagonal matrices $D_1$ and $D_2$ given by

$$L = \begin{bmatrix} \mathsf{I}_1 & 0 \\ \mathsf{c} & \mathsf{I}_2 \end{bmatrix}, \quad U = \begin{bmatrix} \mathsf{I}_1 & \mathsf{b} \\ 0 & \mathsf{I}_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{I}_2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} \mathsf{I}_1 & 0 \\ 0 & \mathsf{d} \end{bmatrix}. \tag{3.1}$$

The submatrices $\mathsf{I}_1$ and $\mathsf{I}_2$ are the identity matrices whose size depends on the configuration and size of the computational grid. Note that the four factors in (3.1) are the most basic units and hence further factorization of any units yields redundant results. Hereinafter, the individual block matrices obtained by exact factorization shall be referred to as the *factorization modules* or *modules* in short. The lower-left block $\mathsf{c}$ in a lower-triangular matrix $L$ involves the divergence operation on velocity and the upper-right block $\mathsf{b}$ in an upper-triangular matrix $U$ involves the velocity update. Because the velocity update cannot be carried out prior to the computation of the pressure in any fractional-step methods, a lower-triangular matrix must always precede an upper-triangular matrix. Hence, an implementable splitting method consists of a factorization of $LU$ type only, e.g., $L D_1 D_2 U$, $L D_2 U D_1$, and $D_1 L D_2 U$ (but not $U D_1 D_2 L$). This is certainly true of the example shown in (2.12).

In order for the block-factorization to be meaningful, the four submatrices $\mathsf{a}$, $\mathsf{b}$, $\mathsf{c}$, and $\mathsf{d}$ must satisfy the nontriviality condition:

$$\mathsf{a} \neq 0, \quad \mathsf{b} \neq 0, \quad \mathsf{c} \neq 0, \quad \mathsf{d} \neq 0. \tag{3.2}$$

The block-diagonal submatrices $\mathsf{a}$ and $\mathsf{d}$ in the block-diagonal modules $D_1$ and $D_2$ are given by

$$\mathsf{a} = \mathsf{A}, \quad \mathsf{d} = -\Delta t \mathsf{D} \mathsf{A}^{-1} \mathsf{G}, \tag{3.3}$$

respectively. However, the submatrices $\mathsf{b}$ and $\mathsf{c}$ in the block-triangular modules take different forms, depending on where they are located in individual factorizations (see below).

It has been found that there are 24 possible combinations of four-part factorizations. Among the 24 combinations, 12 do not satisfy the nontriviality condition (3.2) and therefore these trivial combinations must be discarded. The remaining 12 factorizations are

$$\widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_2 \cdot \widehat{\mathsf{m}}_3 \cdot \widehat{\mathsf{m}}_4 = \begin{cases} LD_1D_2U, & LD_2D_1U, & LD_1UD_2, & LD_2UD_1, \\ LUD_1D_2, & LUD_2D_1, & D_1LUD_2, & D_2LUD_1, \\ D_1LD_2U, & D_2LD_1U, & D_1D_2LU, & D_2D_1LU. \end{cases} \tag{3.4}$$

For the 12 remaining combinations, the submatrices $\mathsf{b}$ and $\mathsf{c}$ can take one of the 4 respective forms:

$$\mathsf{b} = \begin{cases} \Delta t\mathsf{G} & \text{2 occurrences,} \\ \Delta t\mathsf{A}^{-1}\mathsf{G} & \text{6 occurrences,} \\ -\mathsf{D}^{-1} & \text{2 occurrences,} \\ -\mathsf{A}\mathsf{D}^{-1} & \text{2 occurrences;} \end{cases} \qquad \mathsf{c} = \begin{cases} \mathsf{D} & \text{2 occurrences,} \\ \mathsf{D}\mathsf{A}^{-1} & \text{6 occurrences,} \\ -\Delta t^{-1}\mathsf{G}^{-1} & \text{2 occurrences,} \\ -\Delta t^{-1}\mathsf{G}^{-1}\mathsf{A} & \text{2 occurrences.} \end{cases} \tag{3.5}$$

Each of the 12 combinations (3.4) can be used to form exact factorizations of the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ into four, three, and two parts, respectively. Note that only the submatrix $\mathsf{A}$ is square and can be inverted; the identity submatrices are trivial. Because the submatrices $\mathsf{D}$ and $\mathsf{G}$ representing the divergence and gradient operators, respectively, cannot be inverted, those factorizations involving inverses of the divergence or gradient operators are discarded. When the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ is factored into three or two parts, submatrices having inverses of the divergence or gradient operators may or may not disappear. The procedure and results of the four-, three-, and two-part exact block-factorizations are summarized below.

### 3.2. *Results of Exact Block-Factorization*

*Exact block-factorizations into four parts.* When the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ is factored into four parts, it can be written as

$$\mathsf{E}_1^{(4)}\mathsf{E}_2^{(4)}\mathsf{E}_3^{(4)}\mathsf{E}_4^{(4)} = \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_2 \cdot \widehat{\mathsf{m}}_3 \cdot \widehat{\mathsf{m}}_4, \tag{3.6}$$

where $\mathsf{E}_\alpha^{(s)}$ denotes the $\alpha$th factor in an $s$-part factorization and $\widehat{\mathsf{m}}_\alpha$ denotes the $\alpha$th module that can be selected from one of the 12 combinations in (3.4). Discarding 8 factorizations that involve the inverse of the divergence or gradient operator, we find 4 legitimate factorizations with four parts:

$$\mathsf{E}_1^{(4)}\mathsf{E}_2^{(4)}\mathsf{E}_3^{(4)}\mathsf{E}_4^{(4)} = \begin{cases} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}. \end{cases} \tag{3.7}$$

*Exact block-factorizations into three parts.* A three-part block-factorization of the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ can be constructed by combining any two adjacent factorization modules from a combination in (3.4). For each of the 12 combinations, there are three ways to form three-part exact factorizations,

$$\mathsf{E}_1^{(3)}\mathsf{E}_2^{(3)}\mathsf{E}_3^{(3)} = \begin{cases} \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_2 \cdot \widehat{\mathsf{m}}_{34}, \\ \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_{23} \cdot \widehat{\mathsf{m}}_4, \\ \widehat{\mathsf{m}}_{12} \cdot \widehat{\mathsf{m}}_3 \cdot \widehat{\mathsf{m}}_4, \end{cases} \tag{3.8}$$

where

$$\widehat{\mathsf{m}}_{i_1 \cdots i_k} = \prod_{j=i_1}^{i_k} \widehat{\mathsf{m}}_j \tag{3.9}$$

denotes a factor consisting of $k$ adjacent modules (with consecutive indices $i_1, \ldots, i_k$) from one of the 12 possible choices in (3.4); for instance, $\widehat{\mathsf{m}}_{23} = \widehat{\mathsf{m}}_2\widehat{\mathsf{m}}_3$. Because there are three ways to form a three-part factorization, a total of 36 combinations of three-part factorizations are possible. Among the 36 possibilities, 9 factorizations contain submatrices involving $\mathsf{D}^{-1}$ and 9 involve $\mathsf{G}^{-1}$, leaving 18 legitimate three-part factorizations. Among the remaining 18 factorizations, only 9 are independent:

$$\mathsf{E}_1^{(3)}\mathsf{E}_2^{(3)}\mathsf{E}_3^{(3)} = \begin{cases} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix} \\ \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{A} & \Delta t \mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{G} \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{I} & \Delta t \mathsf{A}^{-1}\mathsf{G} \\ 0 & -\Delta t \mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}. \end{cases} \tag{3.10}$$

*Exact block factorizations into two parts.* There are three ways to form two-part exact factorizations for each of the 12 combinations in (3.4):

$$\mathsf{E}_1^{(2)}\mathsf{E}_2^{(2)} = \begin{cases} \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_{234}, \\ \widehat{\mathsf{m}}_{12} \cdot \widehat{\mathsf{m}}_{34}, \\ \widehat{\mathsf{m}}_{123} \cdot \widehat{\mathsf{m}}_4. \end{cases} \tag{3.11}$$

Thirty-six combinations of two-part factorizations can be obtained in this manner, among which 26 combinations are legitimate since 5 factorizations contain submatrices involving $\mathsf{D}^{-1}$ and the other 5 involve $\mathsf{G}^{-1}$. Among the 26 legitimate factorizations, only 6 are independent. The 6 independent block-factorizations with two parts are given by

$$\mathsf{E}_1^{(2)}\mathsf{E}_2^{(2)} = \begin{cases} \begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{A}^{-1}\mathsf{G} \\ \mathsf{D} & 0 \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}\mathsf{A}^{-1} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ \mathsf{D}\mathsf{A}^{-1} & 0 \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}. \end{cases} \tag{3.12}$$

## 4. CONSTRUCTION OF SECOND-ORDER FRACTIONAL-STEP METHODS

### 4.1. *Method of Approximate Block-Factorization*

The exact block-factorizations (3.7), (3.10), and (3.12) of the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ contain the three terms $-\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G}$, $\Delta t\mathsf{A}^{-1}\mathsf{G}$, and $\mathsf{D}\mathsf{A}^{-1}$, which involve the inverse of the submatrix $\mathsf{A}$. Because it often takes a tremendous amount of computer time and cost to obtain the inverse $\mathsf{A}^{-1}$ numerically, it is desirable to approximate the three terms $-\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G}$, $\Delta t\mathsf{A}^{-1}\mathsf{G}$, and $\mathsf{D}\mathsf{A}^{-1}$ instead, except in the case of a fully explicit method. The purpose is to obtain approximations of the three terms to the accuracy of the time-advancement method, thereby yielding consistent splitting methods. We first expand the inverse $\mathsf{A}^{-1}$ about $\mathsf{I}$ for small values of $\Delta t \ll 1$:

$$\mathsf{A}^{-1} = \mathsf{I} + \Delta t\mathsf{B} + \Delta t^2\mathsf{B}^2 + \cdots = \mathsf{I} + \sum_{j=1}^{\infty} \Delta t^j \mathsf{B}^j. \tag{4.1}$$

When a method of $r$th-order accuracy is used, the approximations to the three terms are given by

$$-\Delta t\mathsf{D}\mathsf{A}^{-1}\mathsf{G} = -\Delta t\mathsf{D}(\mathsf{I} + \cdots + \Delta t^{r-1}\mathsf{B}^{r-1})\mathsf{G} + O(\Delta t^{r+1}), \tag{4.2a}$$

$$\Delta t\mathsf{A}^{-1}\mathsf{G} = \Delta t(\mathsf{I} + \cdots + \Delta t^{r-1}\mathsf{B}^{r-1})\mathsf{G} + O(\Delta t^{r+1}), \tag{4.2b}$$

$$\mathsf{D}\mathsf{A}^{-1} = \mathsf{D}(\mathsf{I} + \cdots + \Delta t^{r}\mathsf{B}^{r}) + O(\Delta t^{r+1}). \tag{4.2c}$$

Since fractional-step methods having second-order accuracy are considered in this work, the factored modules are approximated to second order in $\Delta t$ (with $r = 2$).

The next steps toward construction of fractional-step methods involve elimination of the factorizations that cannot be implemented in practice and identification of those that

are either degenerate to schemes with fewer factored parts or equivalent to other schemes with the same number of factored parts. It has been found that only three factorizations are nondegenerate. Others are either degenerate or equivalent to one of these three or else cannot be implemented. By "degeneracy," it is meant that when trivial operations in a method are eliminated (e.g., those involving the trivial identity matrices), the method carries out exactly the same operations in another method with fewer parts. By "equivalence," it is meant that the methods implement the identical operations in an identical sequence. Fractional-step methods constructed from these nondegenerate factorizations are referred to as the *canonical fractional-step methods* hereinafter.

### 4.2. *Second-Order Fractional-Step Methods*

*Two-part fractional-step methods.* Substituting the approximations (4.2) with $r = 2$ into the two-part exact factorizations (3.12) yields fractional-step methods with two parts, which can be written as

$$\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)} = \begin{cases} \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_{234}, \\ \widehat{\mathsf{m}}_{12} \cdot \widehat{\mathsf{m}}_{34}, \\ \widehat{\mathsf{m}}_{123} \cdot \widehat{\mathsf{m}}_4, \end{cases} \tag{4.3}$$

where $\mathbf{\Gamma}_\alpha^{(s)}$is the $\alpha$th part of a splitting method with $s$ parts and

$$\widehat{\mathsf{m}}_{i_1 \cdots i_k} = \widehat{\mathsf{m}}_{i_1 \cdots i_k} + O(\Delta t^2) \tag{4.4}$$

represents the second-order approximation of $\widehat{\mathsf{m}}_{i_1 \cdots i_k}$ with an error $O(\Delta t^2)$; e.g., $\widehat{\mathsf{m}}_{12}$ is the second-order approximation of $\widehat{\mathsf{m}}_{12}$. Note that $\widehat{\mathsf{m}}_{i_1 \cdots i_k} \neq \prod_{j=i_1}^{i_k} \widehat{\mathsf{m}}_j$ in general. The result is summarized in Table I.

The six fractional-step methods can be grouped into two types according to which operator in the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ of the Navier–Stokes equations (2.7) remains the same in the approximated system matrix $\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)}$ of the two-part methods. The approximated system

**TABLE I**
**Second-Order Factorization of System Matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ into Two Parts**

| Code | Factorization, $\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)}$ | Degen/equiv |
|---|---|---|
| D2A | $\begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | Nondegenerate |
| D2B | $\begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}$ | $\Leftrightarrow$ D2A |
| D2C | $\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ \mathsf{D} & 0 \end{bmatrix}$ | Unimplementable |
| P2A | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} + \Delta t\mathsf{DB} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | Nondegenerate |
| P2B | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} + \Delta t\mathsf{DB} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}$ | $\Leftrightarrow$ P2A |
| P2C | $\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ \mathsf{D} + \Delta t\mathsf{DB} & 0 \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}$ | Unimplementable |

*Note.* $\mathbf{\Gamma}_\alpha^{(s)}$ is the $\alpha$th part of a splitting method with $s$ parts; $\Leftrightarrow$, equivalent to.

matrix $\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)}$ of D2A, D2B and D2C is given by

$$\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} - \Delta t^2\mathsf{BG} \\ \mathsf{D} & 0 \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{D}}. \tag{4.5}$$

Comparison with the system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ indicates that the pressure-gradient operator $\Delta t\mathsf{G}$ of $\mathbf{\Gamma}_{\mathrm{NS}}$ has been approximated by $\Delta t\mathsf{AG} = \Delta t\mathsf{G} - \Delta t^2\mathsf{BG} = \Delta t\mathsf{G} + O(\Delta t^2)$, but the divergence operator $\mathsf{D}$ remains unchanged. These methods are said to be of type D (for "divergence"). The system matrix of P2A and P2B is given by

$$\mathbf{\Gamma}_1^{(2)}\mathbf{\Gamma}_2^{(2)} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ \mathsf{D} - \Delta t^2\mathsf{DB}^2 & \Delta t^2\mathsf{DBG} \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{P}}, \tag{4.6}$$

from which one finds that the divergence operator $\mathsf{D}$ of the original system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ has been approximated by $\mathsf{D} - \Delta t^2\mathsf{DB}^2 = \mathsf{D} + O(\Delta t^2)$ (or more properly the continuity equation has been approximated), while the pressure-gradient term remains the same. (The system matrix of D2C is slightly different.) These methods shall be referred to as methods of type P (for "pressure").

Not all the algorithms are implementable in practice, however. For instance, in the case of D2C, the pressure $\phi^{(n+1)}$ is not computed anywhere in the method and hence the velocity $\mathbf{u}^{(n+1)}$ cannot be updated, although the scheme can be "interpreted" as working like D2A. It can be shown that P2C also cannot be implemented. All other methods are equivalent to either D2A or P2A (see Table I). Fractional-step methods D2A and P2A are called the canonical methods of type D and P, respectively.

*Three-part fractional-step methods.* Fractional-step methods with three parts,

$$\mathbf{\Gamma}_1^{(3)}\mathbf{\Gamma}_2^{(3)}\mathbf{\Gamma}_3^{(3)} = \begin{cases} \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_2 \cdot \widehat{\mathsf{m}}_{34}, \\ \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_{23} \cdot \widehat{\mathsf{m}}_4, \\ \widehat{\mathsf{m}}_{12} \cdot \widehat{\mathsf{m}}_3 \cdot \widehat{\mathsf{m}}_4, \end{cases} \tag{4.7}$$

can be obtained by substituting the approximations (4.2) with $r = 2$ into the corresponding exact factorizations (3.10). Table II lists the resulting nine methods.

The nine methods can be grouped into three types according to the system matrix of the methods. The system matrix of D3A, D3B, and D3C is given by

$$\mathbf{\Gamma}_1^{(3)}\mathbf{\Gamma}_2^{(3)}\mathbf{\Gamma}_3^{(3)} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} - \Delta t^2\mathsf{BG} \\ \mathsf{D} & 0 \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{D}}, \tag{4.8}$$

P3A, P3B, and P3C have the system matrix

$$\mathbf{\Gamma}_1^{(3)}\mathbf{\Gamma}_2^{(3)}\mathbf{\Gamma}_3^{(3)} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ \mathsf{D} - \Delta t^2\mathsf{DB}^2 & \Delta t^2\mathsf{DBG} \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{P}}, \tag{4.9}$$

and M3A, M3B, and M3C have

$$\mathbf{\Gamma}_1^{(3)}\mathbf{\Gamma}_2^{(3)}\mathbf{\Gamma}_3^{(3)} = \begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} - \Delta t^2\mathsf{BG} \\ \mathsf{D} - \Delta t^2\mathsf{DB}^2 & -\Delta t^3\mathsf{DB}^2\mathsf{G} \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{M}}. \tag{4.10}$$

The system matrix $\mathbf{\Gamma}_{\mathrm{M}}$ approximates both the pressure-gradient operator and divergence operator (or the continuity equation) to $O(\Delta t^2)$.

**TABLE II**
**Second-Order Factorization of System Matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ into Three Parts**

| Code | Factorization, $\mathbf{\Gamma}_1^{(3)}\mathbf{\Gamma}_2^{(3)}\mathbf{\Gamma}_3^{(3)}$ | Degen/equiv |
|---|---|---|
| D3A | $\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Rightarrow$ D2A |
| D3B | $\begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Rightarrow$ D2A |
| D3C | $\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}$ | $\rightarrow$ D2A |
| P3A | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{A} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Rightarrow$ P2A |
| P3B | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Rightarrow$ P2A |
| P3C | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Rightarrow$ P2A |
| M3A | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | Nondegenerate |
| M3B | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}$ | $\Leftrightarrow$ M3A |
| M3C | $\begin{bmatrix} \mathsf{I} & 0 \\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & -\Delta t\mathsf{DG} \end{bmatrix}$ | $\Leftrightarrow$ M3A |

*Note.* $\mathbf{\Gamma}_{\alpha}^{(s)}$ is the $\alpha$th part of a splitting method with $s$ parts; $\Rightarrow$, degenerate to; $\rightarrow$, degenerate to by interpretation; $\Leftrightarrow$, equivalent to.

Degeneracy and equivalence of the methods have been examined. As shown in Table II, D3A, D3B, and D3C are either degenerate or interpreted as being degenerate to the canonical method D2A; P3A, P3B, and P3C are degenerate to the canonical method P2A; and M3B and M3C are equivalent to the three-part method M3A. Method M3A is nondegenerate and has the system matrix $\mathbf{\Gamma}_{\mathrm{M}}$, which approximates both the pressure-gradient and divergence operators of the original system matrix $\mathbf{\Gamma}_{\mathrm{NS}}$. It is called a canonical method of type M (after "mixture").

*Four-part fractional-step methods.* Substituting the approximations (4.2) with $r = 2$ into the four-part exact factorizations (3.7) yields fractional-step methods with four parts:

$$\mathbf{\Gamma}_1^{(4)}\mathbf{\Gamma}_2^{(4)}\mathbf{\Gamma}_3^{(4)}\mathbf{\Gamma}_4^{(4)} = \widehat{\mathsf{m}}_1 \cdot \widehat{\mathsf{m}}_2 \cdot \widehat{\mathsf{m}}_3 \cdot \widehat{\mathsf{m}}_4. \tag{4.11}$$

The four schemes are shown in Table III. Method D4 is of type D and degenerates to the canonical method D2A; P4 of type P degenerates to P2A; and M4A and M4B of type M degenerate to M3A. There are no nondegenerate fractional-methods that have four parts.

## 5. ANALYSIS OF THE CANONICAL FRACTIONAL-STEP METHODS

The three canonical methods of type D, P, and M obtained by exact and approximate factorization can be written respectively as

$$\begin{bmatrix} \mathsf{A} & 0 \\ \mathsf{D} & -\Delta t\mathsf{DG} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \Delta t\mathsf{G} \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \mathbf{u}^{(n+1)} \\ \phi^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{5.1a}$$

**TABLE III**
**Second-Order Factorization of System Matrix $\mathbf{\Gamma}_{\mathrm{NS}}$ into Four Parts**

| Code | Factorization, $\mathbf{\Gamma}_1^{(4)}\mathbf{\Gamma}_2^{(4)}\mathbf{\Gamma}_3^{(4)}\mathbf{\Gamma}_4^{(4)}$ | Degen/equiv |
|---|---|---|
| D4 | $\begin{bmatrix}\mathsf{A} & 0\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D} & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & 0\\ 0 & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{I} & \Delta t\mathsf{G}\\ 0 & \mathsf{I}\end{bmatrix}$ | $\Rightarrow$ D2A |
| P4 | $\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & 0\\ 0 & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{I} & \Delta t\mathsf{G}\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{A} & 0\\ 0 & \mathsf{I}\end{bmatrix}$ | $\Rightarrow$ P2A |
| M4A | $\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{A} & 0\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & 0\\ 0 & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{I} & \Delta t\mathsf{G}\\ 0 & \mathsf{I}\end{bmatrix}$ | $\Rightarrow$ M3A |
| M4B | $\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D}+\Delta t\mathsf{DB} & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & 0\\ 0 & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{A} & 0\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & \Delta t\mathsf{G}\\ 0 & \mathsf{I}\end{bmatrix}$ | $\Rightarrow$ M3A |

*Note.* $\mathbf{\Gamma}_\alpha^{(s)}$ is the $\alpha$th part of a splitting method with $s$ parts; $\Rightarrow$, degenerate to.

$$\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D}+\Delta t\mathsf{DB} & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{A} & \Delta t\mathsf{G}\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(n+1)}\\ \phi^{(n+1)}\end{bmatrix}=\begin{bmatrix}\mathbf{r}\\ 0\end{bmatrix}+\begin{bmatrix}\mathbf{b}\\ c\end{bmatrix}, \tag{5.1b}$$

$$\begin{bmatrix}\mathsf{I} & 0\\ \mathsf{D}+\Delta t\mathsf{DB} & -\Delta t\mathsf{DG}\end{bmatrix}\begin{bmatrix}\mathsf{A} & 0\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{I} & \Delta tG\\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(n+1)}\\ \phi^{(n+1)}\end{bmatrix}=\begin{bmatrix}\mathbf{r}\\ 0\end{bmatrix}+\begin{bmatrix}\mathbf{b}\\ c\end{bmatrix}. \tag{5.1c}$$

The final form of a fractional-step method is obtained when the expression in the assembled form (5.1) is split into the corresponding number of equations. Two-part methods of type D and P have one tentative velocity $\mathbf{u}^*$, while a three-part method of type M has two tentative velocities $\mathbf{u}^*$ and $\mathbf{u}^{**}$.

The system matrices of the canonical methods are given by

$$\mathbf{\Gamma}_{\mathrm{D}}=\begin{bmatrix}\mathsf{A} & \Delta t\mathsf{G}-\Delta t^2\mathsf{BG}\\ \mathsf{D} & 0\end{bmatrix}, \tag{5.2a}$$

$$\mathbf{\Gamma}_{\mathrm{P}}=\begin{bmatrix}\mathsf{A} & \Delta t\mathsf{G}\\ \mathsf{D}-\Delta t^2\mathsf{DB}^2 & \Delta t^2\mathsf{DBG}\end{bmatrix}, \tag{5.2b}$$

$$\mathbf{\Gamma}_{\mathrm{M}}=\begin{bmatrix}\mathsf{A} & \Delta t\mathsf{G}-\Delta t^2\mathsf{BG}\\ \mathsf{D}-\Delta t^2\mathsf{DB}^2 & -\Delta t^3\mathsf{DB}^2\mathsf{G}\end{bmatrix}, \tag{5.2c}$$

respectively. Inspection of the system matrices reveals that the pressure $\phi^{(n+1)}$ is always first-order accurate in time, independent of the time-advancement scheme used [10]. This observation should be properly interpreted. Note that the accuracy of terms involving the pressure in the momentum and Poisson equations is always second order, since the pressure $\phi^{(n+1)}$ always appears with $\Delta t$, e.g., $\Delta t\mathsf{G}\phi^{(n+1)}$ and $-\Delta t\mathsf{DG}\phi^{(n+1)}$. Therefore, the accuracy of all the split and assembled equations of the canonical methods are second order, an undoubtedly powerful guarantee provided by approximate factorization. The splitting accuracy of fractional-step methods constructed by *ad hoc* splitting may or may not be the same as the time accuracy.

The splitting structure of the canonical methods is analyzed in what follows with a particular emphasis placed on how the projection is approximated in a method. We shall consider only fully or semi-implicit time-advancement methods (i.e., $\mathsf{B}\neq 0$), because if all

the terms in the momentum equations are treated explicitly (i.e., $\mathsf{B} = 0$ and $\mathsf{A} = \mathsf{I}$), there is no distinction among the three methods. When fully or semi-implicit time-advancement methods are used, the submatrix $\mathsf{A}$ represents the time-advancement of the viscous and/or nonlinear terms.

### 5.1. *Canonical Fractional-Step Methods of Type D*

The fractional-step method of type D has two split parts and can be written in semi-discrete form as

$$\mathbb{A}\mathbf{u}^* = \mathbf{r}, \quad \mathbb{D}\mathbf{u}^* - \Delta t \mathbb{D}\mathbb{G}\phi^{(n+1)} = 0, \tag{5.3a,b}$$

$$\mathbf{u}^{(n+1)} + \Delta t \mathbb{G}\phi^{(n+1)} = \mathbf{u}^*. \tag{5.3c}$$

The "momentum equations" (5.3a) are advanced in time without the pressure-gradient term to yield the tentative velocity $\mathbf{u}^*$. The fictitious pressure $\phi^{(n+1)}$ is computed via the Poisson equation (5.3b) and the velocity $\mathbf{u}^{(n+1)}$ is updated in (5.3c) by adding the gradient of the computed pressure field $\phi^{(n+1)}$. Combining the two equations (5.3b) and (5.3c) yields the continuity equation for the velocity $\mathbf{u}^{(n+1)}$,

$$\mathbb{D}\mathbf{u}^{(n+1)} = \mathbb{D}\big(\mathbf{u}^* - \Delta t \mathbb{G}\phi^{(n+1)}\big) = 0, \tag{5.4}$$

from which one finds that the updated velocity field $\mathbf{u}^{(n+1)}$ is divergence-free; i.e., it is identical to the original continuity equation (2.3b).

Combining the equations (5.3a) and (5.3c) yields the assembled equation

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t \mathbb{A}\mathbb{G}\phi^{(n+1)} = \mathbf{r}. \tag{5.5}$$

Comparison with the corresponding Navier–Stokes equations (2.3a) shows that the fictitious pressure $\phi = \phi^{(n+1)}$ is related to the "actual" pressure $p = p^{(n+1)}$ by

$$\mathbb{G}p = \mathbb{A}\mathbb{G}\phi = \mathbb{G}\phi - \Delta t \mathbb{B}\mathbb{G}\phi, \tag{5.6}$$

or equivalently,

$$\mathbb{G}\phi = \mathbb{G}p + \Delta t \mathbb{B}\mathbb{G}p + O(\Delta t^2). \tag{5.7}$$

The Poisson equation (5.4) states that the divergence-free vector field $\mathbf{u}^{(n+1)}$ is obtained from the tentative velocity field $\mathbf{w}_{\mathrm{D}} = \mathbf{u}^*$ by projecting out the gradient field $\Delta t \mathbb{G}\phi$. The projection taking place in a second-order method of type D is compared with the "exact" projection in the semi-discrete Navier–Stokes equations (2.3) in Fig. 1. This can be interpreted as a second-order-accurate version of the exact projection (or Helmholtz decomposition) in the Navier–Stokes equations proposed earlier by Chorin [4] and Temam [11, 12]. The exact projection in the Navier–Stokes equations (2.3) can be represented by

$$\mathbb{D}\mathbf{u}_{\mathrm{NS}} = \mathbb{D}\big(\mathbf{w}_{\mathrm{NS}} - \Delta t \mathbb{G}p^{(n+1)}\big) = 0, \tag{5.8}$$

where $\mathbf{u}_{\mathrm{NS}}$ denotes the solution of the Navier–Stokes equations and

$$\mathbf{w}_{\mathrm{NS}} = \mathbf{r} + (1 - \mathbb{A})\mathbf{u}_{\mathrm{NS}}. \tag{5.9}$$

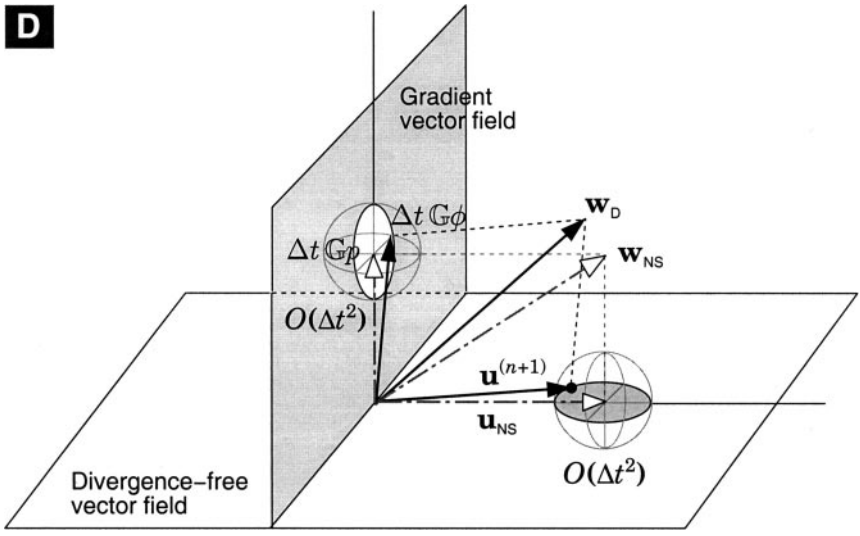


**FIG. 1.** Comparison of the projection in a second-order method of type D with the exact projection in the Navier–Stokes equations: ——▶, second-order method of type D; - - -▷, Navier–Stokes equations. The solution $\mathbf{u}^{(n+1)}$ of this method is exactly divergence-free (within machine roundoff) and has a second-order error $\mathbf{u}^{(n+1)} = \mathbf{u}_{NS} + O(\Delta t^2)$. The associated pressure gradient $\mathbb{G}\phi$ is related to the actual pressure gradient $\mathbb{G}p$ by $\mathbb{G}\phi = \mathbb{G}p + \Delta t\mathbb{B}\mathbb{G}p + O(\Delta t^2)$.

The difference between $\mathbf{w}_D$ and $\mathbf{w}_{NS}$ is given by

$$\mathbf{w}_D - \mathbf{w}_{NS} = \Delta t^2 \mathbb{B}\mathbb{G}p^{(n+1)} + O(\Delta t^3), \tag{5.10}$$

indicating that the velocity field $\mathbf{u}^{(n+1)}$ has a second-order error $O(\Delta t^2)$.

The actual pressure $p^{(n+1)}$ which is not computed during time advancement in this method may be obtained by a separate calculation (5.6). Because the velocity field $\mathbf{u}^{(n+1)}$ is exactly divergence-free in a method of type D, this "divergence-free" method is recommended for computations in which incompressibility of the velocity field is of central interest and/or if the computation of the actual pressure $p$ is not needed during time advancement.

### 5.2. *Canonical Fractional-Step Methods of Type P*

The fractional-step method of type P that has two split parts can be written as

$$\mathbf{u}^* = \mathbf{r}, \quad \mathbb{D}(1 + \Delta t\mathbb{B})\mathbf{u}^* - \Delta t\mathbb{D}\mathbb{G}\phi^{(n+1)} = 0, \tag{5.11a,b}$$

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t\mathbb{G}\phi^{(n+1)} = \mathbf{u}^*. \tag{5.11c}$$

In this method, the "momentum equations" (5.11a) are advanced by an explicit method without the pressure-gradient term and the implicit terms are advanced in the pressure-correction equation (5.11c). Combining these two equations yields the assembled equation

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t\mathbb{G}\phi^{(n+1)} = \mathbf{r}. \tag{5.12}$$

Comparison with the Navier–Stokes equations (2.3a) reveals that the pressure gradient is identical to the "actual" pressure gradient in this method:

$$\mathbb{G}\phi^{(n+1)} = \mathbb{G}p^{(n+1)}. \tag{5.13}$$

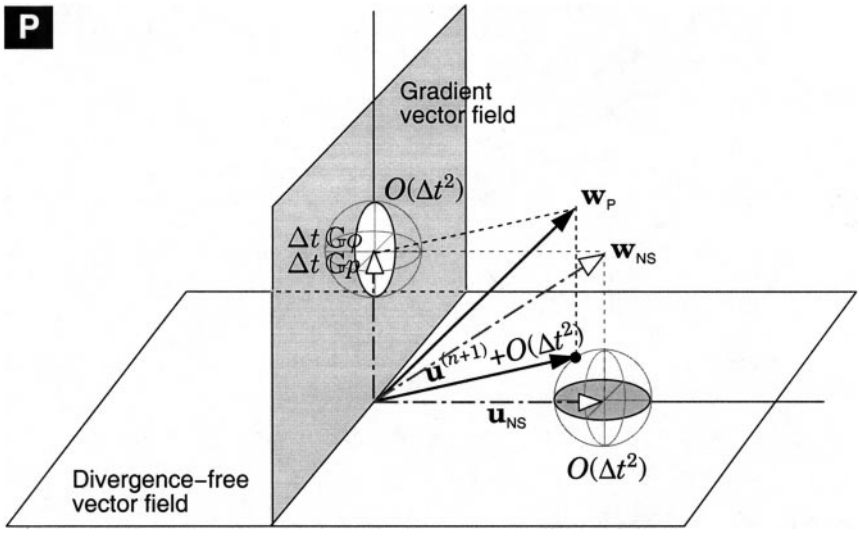


**FIG. 2.** Comparison of the projection in a second-order method of type P with the exact projection in the Navier–Stokes equations: ——▶, second-order method of type P; - - -▷, Navier–Stokes equations. The solution $\mathbf{u}^{(n+1)}$ obtained by this method is approximately divergence-free to $O(\Delta t^2)$ and has a second-order error $\mathbf{u}^{(n+1)} = \mathbf{u}_{\mathrm{NS}} + O(\Delta t^2)$. The associated pressure gradient $\mathbb{G}\phi$ is identical to the actual pressure gradient $\mathbb{G}p$.

The Poisson equation (5.11b) by which the fictitious pressure $\phi^{(n+1)}$ is computed can be regarded as representing the projection of the method (see Fig. 2 for a schematic of the projection). Here,

$$\mathbf{w}_{\mathrm{P}} = \mathbf{u}^* + \Delta t\mathbb{B}\mathbf{u}^* \tag{5.14}$$

is projected into the solenoidal subspace with the pressure gradient $\Delta t\mathbb{G}\phi^{(n+1)}$ separated out into the gradient field. Substituting the pressure-correction equation (5.11c) into the Poisson equation (5.11b) yields

$$\mathbb{D}(1 - \Delta t^2\mathbb{B}^2)\mathbf{u}^{(n+1)} + \Delta t^2\mathbb{D}\mathbb{B}\mathbb{G}\phi^{(n+1)} = 0, \tag{5.15}$$

which indicates that the velocity field $\mathbf{u}^{(n+1)}$ obtained by this method is only approximately divergence-free; the divergence has an $O(\Delta t^2)$ error, unless $\mathbb{D}\mathbb{B}\mathbb{G}\phi^{(n+1)} = \mathbb{D}\mathbb{B}^2\mathbf{u}^{(n+1)}$ is satisfied everywhere in the computational domain. The difference between $\mathbf{w}_{\mathrm{P}}$ and $\mathbf{w}_{\mathrm{NS}}$ is

$$\mathbf{w}_{\mathrm{P}} - \mathbf{w}_{\mathrm{NS}} = -\Delta t^2\left(\mathbb{B}^2\mathbf{u}_{\mathrm{NS}} - \mathbb{G}p^{(n+1)}\right), \tag{5.16}$$

where the relation (5.13) between $\mathbb{G}\phi^{(n+1)}$ and $\mathbb{G}p^{(n+1)}$ has been used.

Despite that the velocity field $\mathbf{u}^{(n+1)}$ is only approximately divergence-free, a method of type P is distinguished by its capability of the "exact" pressure (5.13). Hence, this "pressure-accurate" method is recommended when accurate pressure is preferred to the incompressibility of the velocity field or the actual pressure $p$ should be computed during time advancement.

### 5.3. *Canonical Fractional-Step Methods of Type M*

In fractional-step methods of type M, the pressure gradient operator $\Delta t\mathbb{G}$ and divergence operator $\mathbb{D}$ (or continuity equation) are approximated respectively by $\Delta t\mathbb{G} - \Delta t^2\mathbb{B}\mathbb{G}$ and $\mathbb{D} - \Delta t^2\mathbb{D}\mathbb{B}$ in the system matrix. Writing the three-part method in semidiscrete form, we have

$$\mathbf{u}^* = \mathbf{r}, \quad \mathbb{D}(1 + \Delta t\mathbb{B})\mathbf{u}^* - \Delta t\mathbb{D}\mathbb{G}\phi^{(n+1)} = 0, \tag{5.17a,b}$$

$$\mathbb{A}\mathbf{u}^{**} = \mathbf{u}^*, \tag{5.17c}$$

$$\mathbf{u}^{(n+1)} + \Delta t\mathbb{G}\phi^{(n+1)} = \mathbf{u}^{**}. \tag{5.17d}$$

This method can be considered as a mixture (or hybrid) of methods of type D and P. Assembling the equations (5.17a,c,d) yields

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t\mathbb{A}\mathbb{G}\phi^{(n+1)} = \mathbf{r}, \tag{5.18}$$

which is the same as the assembled equation (5.5) for a method of type D. The fictitious pressure $\phi = \phi^{(n+1)}$ of this method is related to the "actual" pressure $p = p^{(n+1)}$ by

$$\mathbb{G}\phi = \mathbb{G}\phi + \Delta t\mathbb{B}\mathbb{G}p + O(\Delta t^2). \tag{5.19}$$

The "momentum" and Poisson equations (5.17a,b) of this method are identical to those (5.11a,b) of a type-P method. The fictitious pressure $\phi^{(n+1)}$ is computed by solving the Poisson equation (5.17b), which represents the projection of

$$\mathbf{w}_{\mathrm{M}} = \mathbf{u}^* + \Delta t\mathbb{B}\mathbf{u}^* \tag{5.20}$$

into the solenoidal subspace separating the pressure-gradient field $\Delta t\mathbb{G}\phi^{(n+1)}$. Substituting the implicit equation (5.17c) and the velocity-update equation (5.17d) into the Poisson equation (5.17b) yields

$$\mathbb{D}(1 - \Delta t^2\mathbb{B}^2)\mathbf{u}^{(n+1)} - \Delta t^3\mathbb{D}\mathbb{B}^2\mathbb{G}\phi^{(n+1)} = 0, \tag{5.21}$$

showing that the velocity field $\mathbf{u}^{(n+1)}$ is not exactly divergence-free; the divergence has an $O(\Delta t^2)$ error.

## 6. BOUNDARY CONDITIONS FOR THE CANONICAL FRACTIONAL-STEP METHODS

Canonical fractional-step methods of second-order *splitting accuracy* have been constructed based on approximate factorization. In order to obtain second-order-accurate solutions by a fractional-step method, the accuracy of the boundary conditions should match the splitting accuracy as well. The difficulty is that the boundary conditions on the nonphysical quantities are not given *a priori* from the problem formulation and hence have to be determined from the split equations.

In this work, it is shown that the boundary conditions that are consistent with the split equations to the splitting accuracy can be found in a systematic way. The general rule for systematic identification of the consistent boundary conditions is

(i) to approximate the expressions for the tentative velocity and pressure to the splitting accuracy in terms of quantities known at the current time step; and

(ii) to take the limit of the result as the boundary is approached.

The first approximation step involves conversion of implicit terms to the corresponding explicit expressions and the second boundary-limit step may involve extrapolation to the boundary of the data in practice.

### 6.1. *Boundary Condition on the Pressure*

For boundary conditions on pressure, the Neumann type is considered herein, since it is a common practice to use it to compute pressure. It has been widely known that the boundary condition on the pressure is the major source of difficulties when fractional-step methods are considered for computing incompressible flow problems. The greatest difficulty in the determination of the boundary condition on the pressure $\phi^{(n+1)}$ is primarily due to the fact that the natural boundary condition on its counterpart $p^{(n+1)}$ is not available. Therefore, one has to *derive* the approximate expression for the pressure boundary condition in terms of quantities that are available at the current time step.

In order to match the splitting accuracy $O(\Delta t^2)$, the approximate boundary condition on the pressure suffices to be first-order accurate. As Perot [10] correctly pointed out, the first-order accuracy of the pressure $\phi^{(n+1)}$ is intrinsic in the equations of fractional-step methods (see also Section 5). Attempts to improve the order of accuracy of a fractional-step method with an "improved" pressure boundary condition may yield *more accurate results*, which does *not* necessarily lead to solutions of *improved order of accuracy*. This distinction should be made explicit and clear.

The simplest choice to meet the $O(\Delta t)$ requirement for the pressure would be to approximate $\mathbb{G}\phi^{(n+1)}$ in terms of $\mathbb{G}\phi^{(n)}$ to $O(\Delta t)$ since $\phi^{(n)}$ is already known at the current time step. Expanding $\phi^{(n+1)}$ about $\phi^{(n)}$ in time,

$$\phi^{(n+1)} = \phi^{(n)} + \Delta t \frac{\partial \phi^{(n)}}{\partial t} + O(\Delta t^2), \tag{6.1}$$

one finds that it suffices to keep the leading term alone to obtain the second-order-accurate results. By taking the limit of (6.1) as the boundary is approached, the consistent boundary condition on the pressure is obtained,

$$\left[\mathbb{G}_n \phi^{(n+1)}\right]_{\mathrm{B}} = \left[\mathbb{G}_n \phi^{(n)}\right]_{\mathrm{B}} + O(\Delta t), \tag{6.2}$$

where

$$[\cdot]_{\mathrm{B}} = \lim_{\mathbf{x} \to \mathbf{x}_{\mathrm{B}}} [\cdot] \tag{6.3}$$

denotes the limit as the boundary $\mathbf{x}_{\mathrm{B}}$ is approached (boundary limit) and $\mathbb{G}_n = \mathbf{n} \cdot \mathbb{G}$ denotes the gradient in the direction $\mathbf{n}$ normal to the boundary. This boundary condition is referred to as the *current pressure-gradient condition*. The value of the current pressure gradient $\mathbb{G}\phi^{(n)}$ at the boundary may be determined by extrapolating to the boundary the pressure field $\phi^{(n)}(\mathbf{x})$ known at the current time step. Since the pressure boundary condition has been obtained independent of the choice of a fractional-step method, it can be used for any fractional-step methods of second-order accuracy. This pressure boundary condition (6.2) may be considered as the *universal* pressure boundary condition.

An alternative expression for the pressure boundary condition can be obtained from the assembled equations (5.12), (5.5), and (5.18) of methods of type D, P, and M, respectively. Approximation of these equations to $O(\Delta t^2)$ with the term $\Delta t\mathbb{B}\mathbf{u}^{(n+1)}$ replaced by $\Delta t\mathbb{B}\mathbf{u}^{(n)} + O(\Delta t^2)$ yields the same expression for the pressure gradient, namely,

$$\mathbb{G}\phi^{(n+1)} = \frac{1}{\Delta t}\left(\mathbf{r} - \mathbf{u}^{(n+1)}\right) + \mathbb{B}\mathbf{u}^{(n)} + O(\Delta t), \tag{6.4}$$

because only first-order accuracy is required for the pressure. It should be pointed out that the implicit term $\mathbb{A}\mathbf{u}^{(n+1)}$ in the assembled equation has been converted to explicit equivalents by the above approximation. Taking the limit to the boundary gives the alternative expression for the pressure boundary condition,

$$\left[\mathbb{G}_n\phi^{(n+1)}\right]_{\mathrm{B}} = \frac{1}{\Delta t}\left([\mathbf{n}\cdot\mathbf{r}]_{\mathrm{B}} - \mathbf{n}\cdot\mathbf{u}_{\mathrm{B}}^{(n+1)}\right) + \left[\mathbf{n}\cdot\mathbb{B}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t), \tag{6.5}$$

where $\mathbf{u}_{\mathrm{B}}^{(n+1)}$ is the value given by the natural boundary condition (2.2). The boundary condition can be evaluated by extrapolating the data $\mathbf{r}$ and $\mathbb{B}\mathbf{u}^{(n)}$ to the boundary. This alternative expression for the pressure boundary condition agrees with the current pressure gradient condition (6.2) to $O(\Delta t)$.

It is apparent that the zero-pressure-gradient condition

$$\left[\mathbb{G}_n\phi^{(n+1)}\right]_{\mathrm{B}} = 0 + O(1) \tag{6.6}$$

of the marker-and-cell (MAC) method (Harlow and Welch [6]) is zeroth-order accurate and therefore will produce a velocity field $\mathbf{u}^{(n+1)}$ of only first-order accuracy (see for example Armfield and Street [1]).

### 6.2. *Boundary Conditions on the Tentative Velocities*

The boundary conditions on the tentative velocities should be determined from the split equations and hence are method specific.

*Boundary conditions for fractional-step methods of type D.* In order to solve the momentum and Poisson equations (5.3a,b) of a method of type D, the boundary condition on the tentative velocity $\mathbf{u}^*$ should be specified because solving the corresponding algebraic equations (5.1a) involves inversion of the matrices $\mathsf{A}$ and $\mathsf{DG}$. However, the update of the velocity $\mathbf{u}^{(n+1)}$ in (5.3c) involves assignment of data at the interior gridpoints only, for which no boundary conditions are required. The boundary condition on the tentative velocity $\mathbf{u}^*$ of this method can be found from either of the split equations (5.3a,c); the two should give the same result to $O(\Delta t^2)$. The former equation (5.3a) gives

$$\mathbf{u}^* - \Delta t\mathbb{B}\mathbf{u}^* = \mathbf{r}, \tag{6.7}$$

which under the approximation to $O(\Delta t^2)$ reduces to

$$\mathbf{u}^* = \mathbf{r} + \Delta t\mathbb{B}\mathbf{u}^{(n)} + O(\Delta t^2). \tag{6.8}$$

The second-order boundary condition on the tentative velocity is given by

$$[\mathbf{u}^*]_{\mathrm{B}} = [\mathbf{r}]_{\mathrm{B}} + \Delta t\left[\mathbb{B}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2). \tag{6.9}$$

An alternative yet equivalent boundary condition on $\mathbf{u}^*$ can be derived from the latter equation (5.3c) in a simpler way. Approximating the pressure gradient to $O(\Delta t)$ using the expansion (6.1) and taking the limit to the boundary yields

$$[\mathbf{u}^*]_B = \mathbf{u}_B^{(n+1)} + \Delta t \left[\mathbb{G}\phi^{(n)}\right]_B + O(\Delta t^2), \tag{6.10}$$

which should agree with the equivalent expression (6.9) to $O(\Delta t^2)$. Kim and Moin [8] used a different approach to arrive at the same result.

*Boundary conditions for fractional-step methods of type P.* In this method, the boundary condition on $\mathbf{u}^*$ needs to be determined for solution of the coupled equations (5.11b,c). The boundary condition can be found by examining either of the split equations (5.11a,c) by the same procedure explained for methods of type D. From the former equation (5.11a), it immediately follows that

$$[\mathbf{u}^*]_B = [\mathbf{r}]_B + O(\Delta t^2). \tag{6.11}$$

Approximating the latter equation (5.11c) to $O(\Delta t^2)$ and taking the boundary limit of the result gives the alternative expression:

$$[\mathbf{u}^*]_B = \mathbf{u}_B^{(n+1)} + \Delta t \left[\mathbb{G}\phi^{(n)} - \mathbb{B}\mathbf{u}^{(n)}\right]_B + O(\Delta t^2). \tag{6.12}$$

The boundary condition on the tentative velocity of methods D and P differs by an amount $\Delta t[\mathbb{B}\mathbf{u}^{(n)}]_B$, which reflects the difference in splitting between the two methods. This explicit term derives from approximation of its implicit equivalent $\Delta t\mathbb{B}\mathbf{u}^{(n+1)}$ in the operator $\mathbb{A}\mathbf{u}^{(n+1)}$. It would be of interest to investigate the effect of the implicit-to-explicit conversion on stability of the solutions.

*Boundary conditions for fractional-step methods of type M.* The method of type M consists of three split parts and, therefore, there are two tentative velocities $\mathbf{u}^*$ and $\mathbf{u}^{**}$. The solution of Eqs. (5.17b,c) requires the boundary values of $\mathbf{u}^*$ and $\mathbf{u}^{**}$. The boundary conditions on the two tentative velocities can be found by examining the split equations (5.17a,d). The first equation (5.17a) immediately gives the boundary condition on the first tentative velocity $\mathbf{u}^*$:

$$[\mathbf{u}^*]_B = [\mathbf{r}]_B + O(\Delta t^2). \tag{6.13}$$

Approximating the last equation (5.17d) to $O(\Delta t^2)$ and taking the boundary limit of the result yields the boundary condition on the second tentative velocity $\mathbf{u}^{**}$:

$$[\mathbf{u}^{**}]_B = \mathbf{u}_B^{(n+1)} + \Delta t \left[\mathbb{G}\phi^{(n)}\right]_B + O(\Delta t^2). \tag{6.14}$$

The boundary conditions (6.13) and (6.14) are identical to those of methods of type P and D, respectively, reminiscent of the mixture of the two methods.

The consistent boundary conditions on the tentative velocities of methods of type D, P, and M are summarized in Table IV.

**TABLE IV**
**Consistent Second-Order Boundary Conditions for Canonical Fractional-Step Methods of Type D, P, and M**

| Type | $\mathbb{G}_n\phi^{(n+1)}$ | $\mathbf{u}^*$ | $\mathbf{u}^{**}$ |
|---|---|---|---|
| D | $\mathbb{G}_n\phi^{(n)}$ or $\mathbf{n}\cdot[\Delta t^{-1}(\mathbf{r}-\mathbf{u}^{(n+1)})+\mathbb{B}\mathbf{u}^{(n)}]$ | $\mathbf{u}^{(n+1)}+\Delta t\mathbb{G}\phi^{(n)}$ or $\mathbf{r}+\Delta t\mathbb{B}\mathbf{u}^{(n)}$ | — |
| P | $\mathbb{G}_n\phi^{(n)}$ or $\mathbf{n}\cdot[\Delta t^{-1}(\mathbf{r}-\mathbf{u}^{(n+1)})+\mathbb{B}\mathbf{u}^{(n)}]$ | $\mathbf{u}^{(n+1)}+\Delta t(\mathbb{G}\phi^{(n)}-\mathbb{B}\mathbf{u}^{(n)})$ or $\mathbf{r}$ | — |
| M | $\mathbb{G}_n\phi^{(n)}$ or $\mathbf{n}\cdot[\Delta t^{-1}(\mathbf{r}-\mathbf{u}^{(n+1)})+\mathbb{B}\mathbf{u}^{(n)}]$ | $\mathbf{u}^{(n+1)}+\Delta t(\mathbb{G}\phi^{(n)}-\mathbb{B}\mathbf{u}^{(n)})$ or $\mathbf{r}$ | $\mathbf{u}^{(n+1)}+\Delta t\mathbb{G}\phi^{(n)}$ or $\mathbf{r}+\Delta t\mathbb{B}\mathbf{u}^{(n)}$ |

*Note*. All expressions are to be evaluated by extrapolating the corresponding data at the interior of the computational domain, except $\mathbf{u}^{(n+1)}$ for which the natural boundary condition is imposed.

### 6.3. *Boundary Conditions for a Flow with a Steady Zero-Velocity Wall*

Because many standard flow cases consist of steady zero-velocity walls,

$$\mathbf{u}_{\mathrm{B}}^{(n+1)}=\mathbf{u}_{\mathrm{B}}^{(n)}=\cdots=0, \tag{6.15}$$

it is of interest to examine the boundary conditions on the pressure and tentative velocities that can be used in computation of these flows. A stationary no-slip wall is a typical example of such a boundary condition. In this case, the nonlinear terms vanish at the boundary and hence only the viscous terms appear in the boundary conditions on the tentative velocities.

Suppose that the second-order implicit Crank–Nicolson method is used for the viscous terms and an arbitrary second-order explicit method is used for the nonlinear terms: $\mathbb{B}=\frac{1}{2}\mathrm{Re}^{-1}\mathbb{L}$, where $\mathbb{L}$ is the Laplacian operator. Hereinafter, this time-advancement method shall be referred to as the AXCN method. The boundary conditions for the methods of type D, P, and M are compared in Table V.

The alternative pressure boundary condition (6.5) reduces to

$$\left[\mathbb{G}_n\phi^{(n+1)}\right]_{\mathrm{B}}=\frac{1}{\mathrm{Re}}\left[\mathbf{n}\cdot\mathbb{L}\mathbf{u}^{(n)}\right]_{\mathrm{B}}+O(\Delta t). \tag{6.16}$$

**TABLE V**
**Second-Order Boundary Conditions for Canonical Methods: Flow with Steady Zero-Velocity Wall**

| Type | $\mathbb{G}_n\phi^{(n+1)}$ | $\mathbf{u}^*$ | $\mathbf{u}^{**}$ |
|---|---|---|---|
| D | $\mathbb{G}_n\phi^{(n)}$ or $\frac{1}{\mathrm{Re}}\mathbf{n}\cdot\mathbb{L}\mathbf{u}^{(n)}$ | $\frac{\Delta t}{\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ | — |
| P | $\mathbb{G}_n\phi^{(n)}$ or $\frac{1}{\mathrm{Re}}\mathbf{n}\cdot\mathbb{L}\mathbf{u}^{(n)}$ | $\frac{\Delta t}{2\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ | — |
| M | $\mathbb{G}_n\phi^{(n)}$ or $\frac{1}{\mathrm{Re}}\mathbf{n}\cdot\mathbb{L}\mathbf{u}^{(n)}$ | $\frac{\Delta t}{2\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ | $\frac{\Delta t}{\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ |

*Note.* The nonlinear terms are treated explicitly and viscous terms are treated by the Crank–Nicolson method (AXCN method).

This result could have been obtained from the unsplit momentum equations (2.3) with $\mathbb{G}\phi^{(n+1)} = \mathbb{G}p^{(n+1)} + O(\Delta t)$ (see Orszag, Israeli, and Deville [9]; Karniadakis, Israeli, and Orszag [7]).

The boundary conditions on the tentative velocities depend on the method. A method of type D has

$$[\mathbf{u}^*]_{\mathrm{B}} = \frac{\Delta t}{\mathrm{Re}}\left[\mathbb{L}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2). \tag{6.17}$$

The boundary condition for a method of type P is

$$[\mathbf{u}^*]_{\mathrm{B}} = \frac{\Delta t}{2\mathrm{Re}}\left[\mathbb{L}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2). \tag{6.18}$$

The boundary conditions on the two tentative velocities of a method M are

$$[\mathbf{u}^*]_{\mathrm{B}} = \frac{\Delta t}{2\mathrm{Re}}\left[\mathbb{L}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2), \tag{6.19a}$$

$$[\mathbf{u}^{**}]_{\mathrm{B}} = \frac{\Delta t}{\mathrm{Re}}\left[\mathbb{L}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2). \tag{6.19b}$$

## 7. FRACTIONAL-STEP METHODS WITH THE HOMOGENEOUS PRESSURE BOUNDARY CONDITION

### 7.1. *New Pressure that Satisfies the Homogeneous Neumann Boundary Condition*

In constructing a numerical method, a transformation of the pressure variable that reduces the pressure boundary condition to the homogeneous one is of great interest if the transformation involves only a quantity already known at the current time step. Otherwise, evaluation of the boundary condition would require extrapolation of data. Because pressure is linear in the Navier–Stokes equations and its time derivative is absent, one is free to subtract (or add) an arbitrary scalar function from pressure but still obtains the same solutions provided that the change is properly incorporated in the equations and the boundary conditions. (The initial condition may also be altered, but this issue does not affect the account given here.) The system matrix of a fractional-step method remains unchanged under such a transformation, because only the right-hand side $\mathbf{r}$ gets modified whose values are available at the current time step.

The procedure to carry out the transformation for a fractional-step method is to define the new pressure $\Phi$ admitting the homogeneous boundary condition and make changes to the right-hand side vector and tentative velocity. Since the approach taken in the present procedure is straightforward and general, existing, or prospective second-order fractional-step methods can be transformed into the proposed form without difficulty. Once a fractional-step method that admits the homogeneous pressure boundary condition (HPBC) is derived, the boundary condition on the new tentative velocity can be found by following the general rule delineated in Section 6. In fact, the boundary condition on the new tentative velocity becomes simpler in general.

Define the new pressure $\Phi^{(n+1)}$ by subtracting a scalar function $\psi$ from the fictitious pressure $\phi^{(n+1)}$,

$$\Phi^{(n+1)} = \phi^{(n+1)} - \psi, \tag{7.1}$$

where $\psi$ is regarded also as a pressure-like scalar variable. We wish to determine $\psi$ such that the new pressure $\Phi^{(n+1)}$ satisfies the homogeneous Neumann boundary condition:

$$\left[\mathbb{G}_n \Phi^{(n+1)}\right]_{\mathrm{B}} = 0. \tag{7.2}$$

Note that the requirement (7.2) for $\Phi^{(n+1)}$ is equivalent to demanding

$$[\mathbb{G}_n \psi]_{\mathrm{B}} = \left[\mathbb{G}_n \phi^{(n+1)}\right]_{\mathrm{B}}, \tag{7.3}$$

from which $\psi$ may be determined. Since $\phi^{(n+1)}$ is not available at the current time step, however, one has to seek an alternative for it. Approximation of $\phi^{(n+1)}$ in terms of $\phi^{(n)}$ would be the simplest alternative. Expanding $\phi^{(n+1)}$ about $\phi^{(n)}$ in time,

$$\phi^{(n+1)} = \phi^{(n)} + \Delta t \frac{\partial \phi^{(n)}}{\partial t} + O(\Delta t^2), \tag{7.4}$$

and keeping the leading term alone for a second-order method, we see that the requirement (7.3) for $\psi$ is approximately satisfied to $O(\Delta t)$ by choosing

$$\psi = \phi^{(n)} + O(\Delta t). \tag{7.5}$$

This choice of $\psi$ in turn determines the new pressure-like variable

$$\Phi^{(n+1)} = \Delta \phi^{(n+1)} = \phi^{(n+1)} - \phi^{(n)}. \tag{7.6}$$

Note that the current pressure-gradient condition (6.2) is in fact equivalent to the homogeneous boundary condition (7.2) on $\Phi^{(n+1)}$. Note also that $\Phi^{(n+1)}$ is of the order $O(\Delta t)$ and is essentially a delta-form formulation of pressure (see Beam and Warming [2]; Warming and Beam [14]). The delta-form pressure, which appeared previously in Van Kan [13] and Dukowicz and Dvinsky [5], is conceptually equivalent. However, the purpose was to enhance the order of accuracy of the pressure-gradient term, a superfluous attempt (see the discussion in Section 6) that turned out to achieve the goal when the homogeneous boundary condition was used, and the issues pertinent to boundary conditions were not addressed.

We shall now examine how the tentative velocity $\mathbf{u}^*$ and right-hand side $\mathbf{r}$ change due to the change in the pressure variable and then derive the boundary condition on the new tentative velocity. The change in $\mathbf{u}^*$ and $\mathbf{r}$ depends on the splitting in the individual fractional-step methods. The new methods derived from methods of type D, P, and M are denoted by methods of type D*, P*, and M*, respectively.

### 7.2. *Fractional-Step Methods with the Homogeneous Pressure Boundary Condition*

*Methods of type D* with the homogeneous pressure boundary condition.* By replacing the pressure $\phi^{(n+1)}$ with the new pressure $\Phi^{(n+1)}$ in the method of type D (5.3), the new fractional-step method can be written as

$$\mathbb{A}\mathbf{u}^*_{\mathrm{new}} = \mathbf{r}_{\mathrm{new}}, \quad \mathbb{D}\mathbf{u}^*_{\mathrm{new}} - \Delta t \mathbb{D}\mathbb{G}\Phi^{(n+1)} = 0, \tag{7.7a,b}$$

$$\mathbf{u}^{(n+1)} + \Delta t \mathbb{G}\Phi^{(n+1)} = \mathbf{u}^*_{\mathrm{new}}. \tag{7.7c}$$

Comparing the velocity-update equation (7.7c) with the original version (5.3c) of the method, one finds that the new tentative velocity is

$$\mathbf{u}^*_{\text{new}} = \mathbf{u}^* - \Delta t \mathbb{G}\phi^{(n)}. \tag{7.8}$$

Likewise, the new right-hand side $\mathbf{r}_{\text{new}}$ is given by

$$\mathbf{r}_{\text{new}} = \mathbf{r} - \Delta t \mathbb{A}\mathbb{G}\phi^{(n)} = \mathbf{r} - \Delta t \mathbb{G}\phi^{(n)} + O(\Delta t^2). \tag{7.9}$$

The new fractional-step method of type D* admitting the homogeneous pressure boundary condition has been obtained as

$$\mathbb{A}\mathbf{u}^* = \mathbf{r} - \Delta t \mathbb{G}\phi^{(n)}, \quad \mathbb{D}\mathbf{u}^* - \Delta t \mathbb{D}\mathbb{G}\Phi^{(n+1)} = 0, \tag{7.10a,b}$$

$$\mathbf{u}^{(n+1)} + \Delta t \mathbb{G}\Phi^{(n+1)} = \mathbf{u}^*, \tag{7.10c}$$

where the notation $\mathbf{u}^*_{\text{new}}$ has been replaced by $\mathbf{u}^*$. This method is conceptually the same as that used by Choi and Moin [3] in a fully implicit computation of a turbulent channel flow.

Figure 3 shows schematic of the projection taking place in the new method (7.10). It is shown that the new tentative velocity

$$\tilde{\mathbf{w}}_{\text{D}} = \mathbf{u}^* = \mathbf{u}^{(n+1)} + \Delta t \mathbb{G}\Phi^{(n+1)} = \mathbf{u}^{(n+1)} + O(\Delta t^2) \tag{7.11}$$

is projected into the divergence-free subspace with the gradient vector $\Delta t \mathbb{G}\Phi^{(n+1)}$ separated out, yielding the solution $\mathbf{u}^{(n+1)}$. Comparison with the projection in the original method (see Fig. 1) reveals that the tentative velocity is much closer, $O(\Delta t^2)$, to the solution $\mathbf{u}^{(n+1)}$ in the new method than in the original, where the difference is of $O(\Delta t)$. This is because in

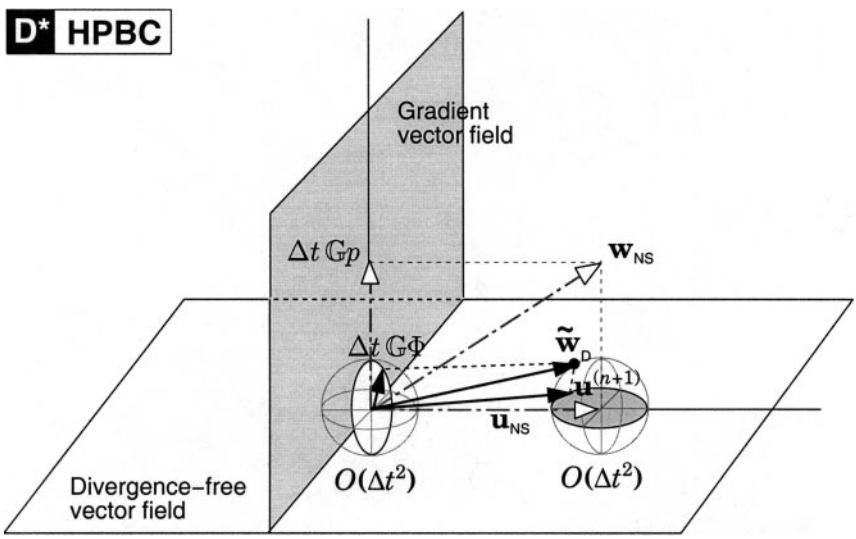


**FIG. 3.** Schematic of the projection in a new second-order method of type D* using the "delta-form" pressure $\Phi^{(n+1)} = \phi^{(n+1)} - \phi^{(n)}$: ——►, present method; ——▷, Navier–Stokes equations. Instead of the usual $\mathbf{w}_{\text{D}} = \mathbf{u}^{(n+1)} + O(\Delta t)$, a new vector field $\tilde{\mathbf{w}}_{\text{D}} = \mathbf{w}_{\text{D}} - \Delta t \mathbb{G}\phi^{(n)} = \mathbf{u}^{(n+1)} + O(\Delta t^2)$ is used for the projection in this method. The solution $\mathbf{u}^{(n+1)}$ is exactly divergence-free (within machine roundoff) and has a second-order error $\mathbf{u}^{(n+1)} = \mathbf{u}_{\text{NS}} + O(\Delta t^2)$. Since the associated pressure gradient is so small, $\mathbb{G}\phi^{(n+1)} = O(\Delta t)$, the homogeneous Neumann boundary condition, $[\mathbb{G}\phi^{(n+1)}]_{\text{B}} = 0$, can be used for second-order computations.

the original method the tentative velocity is obtained from Eq. (5.3) without the pressure-gradient term, whereas in the new method the new tentative velocity $\mathbf{u}^*$ is computed with the pressure-gradient term $\Delta t\mathbb{G}\phi^{(n)}$ that approximates the actual pressure gradient $\Delta t\mathbb{G}p$ to $O(\Delta t^2)$. This salient feature of improved accuracy built in the tentative velocity is the key to arriving at the simple boundary condition for this method as we shall see below.

*Methods of type P* with the homogeneous pressure boundary condition.* By exactly the same procedure for the method of type D*, the new fractional-step method of type P* can be constructed. The new tentative velocity is given by

$$\mathbf{u}^*_{\text{new}} = \mathbf{u}^* - \Delta t\mathbb{G}\phi^{(n)} \tag{7.12}$$

and new right-hand side $\mathbf{r}_{\text{new}}$ is

$$\mathbf{r}_{\text{new}} = \mathbf{r} - \Delta t\mathbb{G}\phi^{(n)}. \tag{7.13}$$

Note that the new right-hand side here is almost the same as that in the method of type P, differing only by $O(\Delta t^2)$. The new fractional-step method of type P* which admits the homogeneous pressure boundary condition is thus obtained as

$$\mathbf{u}^* = \mathbf{r} - \Delta t\mathbb{G}\phi^{(n)}, \quad (\mathbb{D} + \Delta t\mathbb{D}\mathbb{B})\mathbf{u}^* - \Delta t\mathbb{D}\mathbb{G}\Phi^{(n+1)} = 0, \tag{7.14a,b}$$

$$\mathbb{A}\mathbf{u}^{(n+1)} + \Delta t\mathbb{G}\Phi^{(n+1)} = \mathbf{u}^*. \tag{7.14c}$$

*Methods of type M* with the homogeneous pressure boundary condition.* A new fractional-step method of type M* can be constructed by using the same procedure explained above. The new tentative velocities are given by

$$\mathbf{u}^*_{\text{new}} = \mathbf{u}^* - \Delta t\mathbb{G}\phi^{(n)} + O(\Delta t^2), \tag{7.15a}$$

$$\mathbf{u}^{**}_{\text{new}} = \mathbf{u}^{**} - \Delta t\mathbb{G}\phi^{(n)} \tag{7.15b}$$

and new right-hand side $\mathbf{r}_{\text{new}}$ is

$$\mathbf{r}_{\text{new}} = \mathbf{r} - \Delta t\mathbb{G}\phi^{(n)} + O(\Delta t^2). \tag{7.16}$$

Comparison of these new variables with those of methods of type D* and P* indicates the hybrid nature of this method. The new fractional-step method of type M* can be written as

$$\mathbf{u}^* = \mathbf{r} - \Delta t\mathbb{G}\phi^{(n)}, \quad (\mathbb{D} + \Delta t\mathbb{D}\mathbb{B})\mathbf{u}^* - \Delta t\mathbb{D}\mathbb{G}\Phi^{(n+1)} = 0, \tag{7.17a,b}$$

$$\mathbb{A}\mathbf{u}^{**} = \mathbf{u}^*, \tag{7.17c}$$

$$\mathbf{u}^{(n+1)} + \Delta t\mathbb{G}\Phi^{(n+1)} = \mathbf{u}^{**}. \tag{7.17d}$$

### 7.3. *Boundary Conditions on the Tentative Velocities*

Since all tentative velocities $\mathbf{u}^*$ and $\mathbf{u}^{**}$ of the new methods differ respectively from those of the corresponding original methods by $-\Delta t\mathbb{G}\phi^{(n)}$ to $O(\Delta t^2)$, the boundary conditions differ also by the same amount. The boundary conditions for methods of type D*, P*, and M* are summarized in Table VI. Compare the difference with those of the original methods in Table IV.

The boundary condition on $\mathbf{u}^*$ of a method of type D* is given by

$$[\mathbf{u}^*]_{\text{B}} = \mathbf{u}_{\text{B}}^{(n+1)} + O(\Delta t^2). \tag{7.18}$$

**TABLE VI**
**Consistent Second-Order Boundary Conditions for Fractional-Step Methods D*, P*, and M***

| Type | $\mathbb{G}_n\Phi^{(n+1)}$ | $\mathbf{u}^*$ | $\mathbf{u}^{**}$ |
|---|---|---|---|
| D* | 0 | $\mathbf{u}^{(n+1)}$ | — |
| P* | 0 | $\mathbf{u}^{(n+1)} - \Delta t\mathbb{B}\mathbf{u}^{(n)}$ or $\mathbf{r} - \Delta t\mathbb{G}\phi^{(n)}$ | — |
| M* | 0 | $\mathbf{u}^{(n+1)} - \Delta t\mathbb{B}\mathbf{u}^{(n)}$ or $\mathbf{r} - \Delta t\mathbb{G}\phi^{(n)}$ | $\mathbf{u}^{(n+1)}$ |

*Note.* All expressions are to be evaluated by extrapolating the corresponding data at the interior of the computational domain, except $\mathbf{u}^{(n+1)}$ for which the natural boundary condition is imposed.

The boundary condition becomes so simple because the new tentative velocity $\mathbf{u}^*$ is only $O(\Delta t^2)$ away from the velocity $\mathbf{u}^{(n+1)}$, as indicated by (7.11) and in Fig. 3. This same boundary condition is satisfied by the second tentative velocity $\mathbf{u}^{**}$ of a method of type M*.

The boundary condition on $\mathbf{u}^*$ of methods of type P* and M* is given by

$$[\mathbf{u}^*]_{\mathrm{B}} = \begin{cases} [\mathbf{r}]_{\mathrm{B}} - \Delta t\left[\mathbb{G}\phi^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2), & \text{(7.19a)} \\ \mathbf{u}_{\mathrm{B}}^{(n+1)} - \Delta t\left[\mathbb{B}\mathbf{u}^{(n)}\right]_{\mathrm{B}} + O(\Delta t^2). & \text{(7.19b)} \end{cases}$$

### 7.4. *Boundary Conditions for a Flow with a Steady Zero-Velocity Wall*

The boundary conditions in the special case when the natural boundary condition is imposed by the steady zero velocity (6.15) are considered. The time-advancement used is the semi-implicit AXCN method (see Section 6.3). Table VII shows the summary of the boundary conditions for the tentative velocities of methods of type D*, P*, and M*.

The boundary condition (7.18) on $\mathbf{u}^*$ of a method of type D* becomes homogeneous in this case:

$$[\mathbf{u}^*]_{\mathrm{B}} = 0 + O(\Delta t^2). \tag{7.20}$$

Note that this result does not depend on the time-advancement scheme, whether explicit or implicit. This condition is also satisfied by $\mathbf{u}^{**}$ of a method of type M*. The tentative

**TABLE VII**
**Second-Order Boundary Conditions for Methods D*, P*, and M*: Flow with Steady Zero-Velocity Wall**

| Type | $\mathbb{G}_n\Phi^{(n+1)}$ | $\mathbf{u}^*$ | $\mathbf{u}^{**}$ |
|---|---|---|---|
| D* | 0 | 0 | — |
| P* | 0 | $-\dfrac{\Delta t}{2\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ | — |
| M* | 0 | $-\dfrac{\Delta t}{2\mathrm{Re}}\mathbb{L}\mathbf{u}^{(n)}$ | 0 |

*Note.* The nonlinear terms are treated explicitly and viscous terms are treated by the Crank–Nicolson method (AXCN method).

velocity $\mathbf{u}^*$ of methods of type P* and M* has the boundary condition given by

$$[\mathbf{u}^*]_B = -\frac{\Delta t}{2\mathrm{Re}}\left[\mathbb{L}\mathbf{u}^{(n)}\right]_B + O(\Delta t^2). \tag{7.21}$$

## 8. SUMMARY AND CONCLUDING REMARKS

Second-order fractional-step methods and boundary conditions for the incompressible Navier–Stokes equations have been studied. The present work has been focused on (i) identification and analysis of all possible splitting methods of second-order splitting accuracy and (ii) determination of consistent boundary conditions that yield second-order accurate solutions. The account and results given here do not depend on any particular time-advancement schemes, discretization methods for the spatial derivatives, or the geometry of the computational domain.

In order to construct splitting methods, exact and approximate factorization techniques have been used to split the system matrix of the fully discretized Navier–Stokes equations in the most general way. These splitting methods are always guaranteed to have the accuracy of the time-advancement scheme, thanks to the distinguished property of approximate factorization. It has been found that there are three canonical (independent and nondegenerate) types D, P, and M of splitting methods to which all other second-order splitting schemes are either degenerate or equivalent. In *divergence-free* methods of type D, the pressure-gradient term of the Navier–Stokes equations is approximated to $O(\Delta t^2)$; and in *pressure-accurate* methods of type P, the divergence operator (or continuity equation) is approximated to $O(\Delta t^2)$. Both operators are approximated in a method of type M, which can be regarded as mixture (or hybrid) of methods of type D and P.

The discrete projection that takes place in the canonical methods has been analyzed in the light of Chorin's decomposition idea. In a method of type D, the velocity field $\mathbf{u}^{(n+1)}$ is exactly divergence-free and the "fictitious" pressure $\phi^{(n+1)}$ differs from the "actual" pressure $p^{(n+1)}$ by $O(\Delta t)$, whereas the velocity field $\mathbf{u}^{(n+1)}$ of a method of type P is approximately divergence-free $\nabla \cdot \mathbf{u}^{(n+1)} = O(\Delta t^2)$, but the fictitious pressure is identical to the actual pressure. Therefore, a method of type D is recommended for computations in which the incompressibility of the velocity field is required or preferred to pressure, while a method of type P is better suited to the cases when highly accurate pressure has priority over the incompressibility requirement or the actual pressure should be computed during time advancement.

A systematic procedure to find the consistent boundary conditions on the tentative velocity $\mathbf{u}^*$ and pressure $\phi^{(n+1)}$ has been developed, which consists of approximation of the split equations to the splitting accuracy and the boundary limit of the result. The approximation involves conversion of implicit terms to explicit equivalents and the boundary limit is carried out by extrapolation of known data in practice. The results are summarized in Table IV. The pressure boundary condition suffices to be first-order accurate and attempts with "improved" boundary conditions do not necessarily lead to improvement in the *order* of accuracy of the solutions. It has been found that the current pressure-gradient boundary condition does not depend on the type of fractional-step methods and thus can be regarded as being universal: $[\mathbb{G}_n\phi^{(n+1)}]_B = [\mathbb{G}_n\phi^{(n)}]_B$. The alternative expression for the pressure boundary condition turned out to be also independent of the type of fractional-step methods. It has been shown that the boundary condition on the tentative velocity $\mathbf{u}^*$ of the canonical methods can be

expressed in terms of the natural boundary condition $\mathbf{u}_B^{(n+1)}$ and derivatives of quantities available at the current time step (to be evaluated by extrapolation).

Second-order fractional-step methods D*, P*, and M* that admit the homogeneous pressure boundary condition $[\mathbb{G}_n \Phi^{(n+1)}]_B = 0$ have been derived by using a transformation which involves the delta-form pressure $\Phi^{(n+1)} = \phi^{(n+1)} - \phi^{(n)}$. The system matrices of the methods remain the same under the transformation but the tentative velocity is modified by $\Delta t \mathbb{G} \phi^{(n)}$. The boundary conditions obtained for the new methods are summarized in Table VI. In the new method of type D*, the boundary condition on the tentative velocity $\mathbf{u}^*$ becomes $[\mathbf{u}^*]_B = \mathbf{u}_B^{(n+1)}$, a great simplification due to the improved accuracy built into the new tentative velocity $\mathbf{u}^* = \mathbf{u}^{(n+1)} + O(\Delta t^2)$.

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